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SERIES PARALLEL PRECEDENCE CONSTRAINTS

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ABSTRACT

Suppose that n jobs are to be performed by a single machine subject to precedence constraints that can be characterized as "series parallel". Each job j has associated with it a known processing time $p_j$ and weight $w_j$. The objective is sequence the jobs so as to minimize total weighted completion time. It is shown that this problem can be solved by an algorithm with worst case running time of $O(n \log n)$. This result is implicit in results of Sidney, and algorithms of Baker, Horn, and Adolphson and Hu, which dealt with precedence constraints in the form of rooted trees.

# 1. INTRODUCTION

Suppose n jobs are to be performed by a single machine. Each job j has associated with it a known processing time $p_j > 0$ and weight $w_j$, $j = 1, 2, \ldots \ldots, n$. For any given sequence of jobs, $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$, the completion time of job $\sigma(i)$ is

$$C_{\sigma(i)} = \sum_{k=1}^{i} p_{\sigma(k)},$$

and the total weighted completion time of $\sigma$ is

$$\sum_{i=1}^{n} w_{\sigma(i)} C_{\sigma(i)}.$$

In the absence of further constraints, the problem of finding a sequence which minimizes total weighted completion time is solved by a simple rule due to W.E. SMITH [11]: For each job j, form the ratio

$$\rho_j = \frac{w_j}{p_j},$$

and then place the jobs in monotone nonincreasing ratio order. Clearly, this task can be accomplished in 0(n log n) running time.

Now suppose that precedence constraints are specified, in the form of an (irreflexive) partial order "$\prec$". Jobs i must precede job j if $i \prec j$.

For arbitrary precedence constraints, the problem is quite difficult and may be NP-complete in the sense of COOK and KARP [6], even if all job weights are equal. At the time this is written, NP-completeness remains an open question [9]. (Note that this is in constrast to the problem of minimizing the *maximum* weighted completion time, subject to arbitrary precedence constraints, for which there is an efficient algorithm [7].)

However, if the precedence constraints are suitably restricted, the

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problem is much easier. CONWAY, et al, [4] dealt with precedence constraints in the form of "parallel chains". BAKER [3] and HORN [5] solved the problem for more general precedence constraints in the form of rooted trees (or "arborecences"), either rooted to a point or from a point. ADOLPHSON and HU [1], working in the context of the "optimal linear ordering" problem (cf. Section 6), showed that Horn's algorithm can be implemented in $O(n \log n)$ running time. SIDNEY [10] has obtained a number of theorems concerning arbitrary precedence constraints, but his results do not suggest a polynomial bounded algorithm for the general case.

In this paper, we make use of some of Sidney's results to obtain an algorithm for solving the sequencing problem in the case that the precedence constraints can be characterized as "series parallel". We also show that this algorithm can be implemented with worst case running time of $O(n \log n)$. Since rooted trees are special cases of series parallel constraints, this result provides a proper generalization of previous results.

## 2. SERIES PARALLEL DIGRAPHS

We represent a given set of precedence constraints by a digraph $G = (N,A)$ in the obvious way. That is, each node $j \in N$ corresponds to a job and each arc $(i,j) \in A$ corresponds to a pair of jobs in the relation, i.e. $i \prec j$. We now proceed to define series parallel precedence constraints in terms of digraphs.

The class of *transitive series parallel* digraphs is defined recursively as follows:

(2.1) A digraph consisting of a single node, e.g. $G = (\{i\},\emptyset)$, is transitive series parallel.

(2.2) If $G_1 = (N_1,A_1)$ and $G_2 = (N_2.A_2)$, where $N_1 \cap N_2 = \emptyset$, are transitive series parallel, then

$$G = G_1 \times G_2 = (N_1 \cup N_2, A_1 \cup A_1 \cup N_1 \times N_2)$$

is also transitive series parallel. G is said to be the *series composition* of $G_1$ and $G_2$.

(2.3) If $G_1 = (N_1,A_1)$ and $G_2 = (N_2,A_2)$, where $N_1 \cap N_2 = \emptyset$, are transitive series parallel, then

$$G = G_1 \cup G_2 = (N_1 \cup N_2, A_1 \cup A_2)$$

is also transitive series parallel. G is said to be the *parallel composition* of $G_1$ and $G_2$.

(2.4) Only those digraphs which can be formed by a finite number of applications of rules (2.1) - (2.3) are transitive series parallel.

A digraph G is said to be series parallel if and only if its transitive closure is transitive series parallel. Some examples of series parallel digraphs are shown in Figure 1. Note that every series parallel digraph is acyclic. The simplest acyclic digraph which is not series parallel is shown in Figure 2.

Given a series parallel digraph G, it is possible to repeatedly decompose G into series and parallel components, so as to show how G can be obtained by application of rules (2.1) - (2.3). The result is a rooted binary tree we call a *decomposition tree*. Each leaf of the decomposition tree is identified with a node of G. Each internal node marked "S" indicates the series composition of the subgraphs identified with its sons, with the convention that the left son precedes the right son. Each inter-

nal node marked "P" indicates the parallel composition of the subgraphs

identified with its sons. (Here the left-right ordering of sons is unimpor-

tant.) Decomposition trees $T_1, T_2, T_3$ for the digraphs $G_1, G_2, G_3$ shown in

Figure 1 are given in Figure 3. The "S's" and "P's" in the internal nodes

of $T_3$ are given subscripts to facilitate reference in the next section.

In [8] it is shown how to test a given digraph to determine if it is

series parallel and, if it is, to obtain a decomposition tree. This task

can be accomplished on $O(n^2)$ running time. In the sequel, we shall assume

that a decomposition tree is already known for any given precedence con-

straints. Hence, when the claim of $O(n \log n)$ running time is made for the

algorithm presented in this paper, this claim must be qualified to apply

to a problem for which a decomposition tree is given.


## 3. A RECURSIVE PROCEDURE

Suppose we are able to construct optimal sequences for the series and

parallel compositions of subsets of jobs $N_1, N_2$, given optimal sequences for

$N_1$ and $N_2$. Then we have all that is necessary for a recursive procedure for

solving sequencing problems with series parallel precedence constraints.

Rules for these constructions are as follows.

*Series Composition*

Let $\sigma_1, \sigma_2$ be optimal sequences for $N_1, N_2$. Then an optimal sequence for

the series composition of $N_1$ and $N_2$ is simply $\sigma = \sigma_1 \sigma_2$, the concatenation

of $\sigma_1$ and $\sigma_2$.

The proof of this assertion is trivial and is omitted.

*Parallel Composition*

Let $\sigma_1, \sigma_2$ be optimal sequences for $N_1, N_2$, where

$$\sigma_i = (\sigma_i(1), \sigma_i(2), \ldots, \sigma_i(n_i)), \qquad i = 1, 2,$$

and job $\sigma_i(k)$ has weight $w_{ik}$ and processing time $p_{ik}$. An optimal sequence for the parallel composition of $N_1$ and $N_2$ is obtained by repeatedly finding "maximum-ratio prefixes" of $\sigma_1, \sigma_2$ and placing there in successive positions of the solution sequence $\sigma$. This technique is implemented by the following algorithm.

<u>Step 0</u>. Let $\sigma = \lambda$, the empty sequence. Set $m_i = 1$, $i = 1, 2$.

<u>Step 1</u>. If $m_i = n_i + 1$, for $i = 1, 2$, stop; $\sigma$ is an optimal sequence for the parallel composition of $N_1, N_2$. Otherwise, continue to Step 2.

<u>Step 2</u>. Compute

$$\rho_{ir} = \frac{\sum\limits_{j=m_i}^{r} w_{ij}}{\sum\limits_{j=m_i}^{r} p_{ij}}, \qquad\qquad i = 1, 2; \ r = m_i, m_i+1, \ldots, n_i.$$

Find

$$\rho_i^* = \max_{m_i \le r \le n_i} \{\rho_{ir}\}, \qquad\qquad i = 1, 2,$$

and values $k, m$ such that

$$\rho_{km} = \max\{\rho_1^*, \rho_2^*\}.$$

Let

$$\sigma_k^* = (\sigma_k(m_k), \sigma_k(m_k+1), \ldots, \sigma_k(m)).$$

Set $\sigma = \sigma\sigma_k^*$, the concatenation of $\sigma$ and $\sigma_k^*$, and set $m_k = m + 1$. Return to Step 1.

This algorithm is essentially the same as Sidney's "parallel chain" algorithm, with some changes in notation. The validity of the algorithm in this context follows immediately from Sidney's theorem 21. We refer the reader to his paper [10] for a proof.

*An Example*

With the above rules for series and parallel composition it is clearly possible to carry out a recursive computation, starting at the bottom of the decomposition tree and working upwards until an optimal sequence is found for the complete set of jobs. As an example, consider the problem with precedence constraints given by digraph $G_3$ in Figure 1 and with weights and processing times as follows:

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|
| $w_j$ | 1 | 1 | 3 | 2 | 7 | 2 | 10 | 3 | 1 | 3 | 4 | 9 | 1 |
| $p_j$ | 1 | 4 | 1 | 3 | 2 | 8 | 1 | 5 | 10 | 7 | 8 | 2 | 6 |

Optimal solutions for the various subproblems, defined by the decomposition tree $T_3$ in Figure 3, are as follows:

$$P_1 : \sigma_1 = (7,8)$$

$$S_2 : \sigma_2 = (7,8,10)$$

$$S_3 : \sigma_3 = (4,7,8,10)$$

$$P_4 : \sigma_4 = (3,2)$$

$$S_5 : \sigma_5 = (5,9)$$

$$P_6 : \sigma_6 = (5,6,9)$$

$$S_7 : \sigma_7 = (5,6,9,11)$$

$$S_8 : \sigma_8 = (3,2,5,6,9,11)$$

$$P_9 : \sigma_9 = (4,7,3,2,5,8,10,6,9,11)$$

$$P_{10}: \sigma_{10} = (12,13)$$

$$S_{11}: \sigma_{11} = (4,7,3,2,5,8,10,6,9,11,12,13)$$

$$S_{12}: \sigma_{12} = (1,4,7,3,2,5,8,10,6,9,11,12,13)$$

As an example of parallel composition, consider in detail the formation of the sequence $\sigma_9$ from the sequences $\sigma_3$ and $\sigma_8$. At successive executions of Step 2 of the algorithm for parallel computation, data are as follows:

| $\sigma_3^*$ | $\rho_3^*$ | $\sigma_8^*$ | $\rho_8^*$ | $\sigma_9$ |
|---|---|---|---|---|
| (4,7) | 12/4 | (3) | 3/1 | $\lambda$ |
| (8) | 3/5 | (3) | 3/1 | (4,7) |
| (8) | 3/5 | (2,5) | 8/6 | (4,7,3) |
| (8) | 3/5 | (6,9,11) | 6/26 | (4,7,3,2,5) |
| (10) | 3/7 | (6,9,11) | 6/26 | (4,7,3,2,5) |
| $\lambda$ | $-\infty$ | (6,9,11) | 6/26 | (4,7,3,2,5,10) |
| | | | | (4,7,3,2,5,10,6,9,11) |

## 4. MODIFICATION OF RECURSIVE PROCEDURE

We now present a number of refinements and modifications of the basic recursive procedure described in the previous section. The principal objective of these modifications is to enable us to represent the solutions to subproblems in the recursion by *sets* rather than *sequences*. It is believed

that the validity of assertions we make in this section is intuitively ob-
vious, and that proofs can safely be left to the reader.

*Composite Jobs*

Suppose that the parallel composition of sequences $\sigma_1$, $\sigma_2$ is carried
out as part of the recursive computation, and that $\sigma_i^*$ is a maximum-ratio
sequence found in Step 2 of the subalgorithm for parallel composition. We
assert that there exists an optimal sequence for the complete set of jobs
in which the jobs in $\sigma_i^*$ are consecutive. Accordingly, it is possible to
treat the subsequence $\sigma_i^*$ as a single *composite job*, with processing time
and weight set equal to the sums of the processing times and weights of the
jobs contained within it.

Moreover, we assert that there is no change in the ultimate outcome
of the computation if maximum-ratio subsequences are found, and composite
jobs formed, at the time of series, rather than parallel, composition. As
a consequence of this modification, the solution to each subproblem can be
represented by a set of jobs, some or all of which are composite. An optimal
sequence for a subproblem can be constructed by simply placing these jobs
in monotonically nonincreasing order of $w_j/p_j$ ratios.

The rules for series and for parallel composition can now be restated
as follows.

*Series Composition*

Suppose that solutions to two subproblems are given by sets $N_1$ and $N_2$
of elementary and composite jobs. The series composition of $N_1$ and $N_2$ can
result in the formation of at most one new composite job. Moreover, this
composite job must contain a minimum ratio job from $N_1$ and a maximum

ratio job from $N_2$. The following algorithm is now used for series composition.

Step 0. Find

$$\rho_1^* = \min_{j \in N_1'} \left\{\frac{w_j}{p_j}\right\},$$

$$\rho_2^* = \max_{j \in N_2'} \left\{\frac{w_j}{p_j}\right\},$$

and $j(1), j(2)$ such that

$$\frac{w_{j(i)}}{p_{j(i)}} = \rho_i^*, \qquad\qquad i = 1,2.$$

If $\rho_1^* \geq \rho_2^*$, then stop; the solution to the series composition of $N_1, N_2$ is given by $N_1' \cup N_2'$ (no composite job is formed). If $\rho_1^* < \rho_2^*$, then form a composite job $k = (j(1), j(2))$, with $w_k = w_{j(1)} + w_{j(2)}$, $p_k = p_{j(1)} + p_{j(2)}$, set $N_i' = N_i' - \{j(i)\}$, $i = 1,2$, and proceed to Step 1.

Step 1. Find $\rho_1^*$, $\rho_2^*$, $j(1)$, $j(2)$, as in Step 0. If $\rho_1^* \geq w_k/p_k \geq \rho_2^*$, then stop; the solution to the series composition of $N_1, N_2$ is given by $N_1' \cup N_2' \cup \{k\}$. Otherwise proceed to Step 2.

Step 2. If $\rho_1^* < w_k/p_k$, then replace $k$ by $j(1), k$ (the concatenation of $j(1)$ and $k$), set $w_k = w_k + w_{j(1)}$, $p_k = p_k + p_{j(1)}$, and $N_1' = N_1' - \{j(1)\}$. If $\rho_1^* \geq w_k/p_k$ and $w_k/p_k < \rho_2^*$, then replace $k$ by $k, j(2)$, set $w_k = w_k + w_{j(2)}$, $p_k = p_k + p_{j(2)}$, and $N_2' = N_2' - \{j(2)\}$. Return to Step 1.

*Parallel Composition*

Suppose that solutions to two subproblems are given by sets $N_1, N_2$ of

elementary and composite jobs. A solution for parallel composition of $N_1$ and $N_2$ is given simply by $N_1 \cup N_2$.

*An Example*

We now rework the example of the previous section with these new procedures. We denote composite jobs by sequences, e.g. $N_8$ contains the elementary job 3 and composite jobs (2,5) and (6,9,11). The reader should trace through the subalgorithm for series composition to verify the formation of composite jobs.

$$P_1 \; : \; N_1 \;\; = \; \{7,8\}$$

$$S_2 \; : \; N_2 \;\; = \; \{7,8,10\}$$

$$S_3 \; : \; N_3 \;\; = \; \{(4,7),8,10\}$$

$$P_4 \; : \; N_4 \;\; = \; \{3,2\}$$

$$S_5 \; : \; N_5 \;\; = \; \{5,9\}$$

$$P_6 \; : \; N_6 \;\; = \; \{5,6,9\}$$

$$S_7 \; : \; N_7 \;\; = \; \{5,(6,9,11)\}$$

$$S_8 \; : \; N_8 \;\; = \; \{3,(2,5),(6,9,11)\}$$

$$P_9 \; : \; N_9 \;\; = \; \{(4,7),3,(2,5),8,10,(6,9,11)\}$$

$$P_{10}: \; N_{10} \; = \; \{12,13\}$$

$$S_{11}: \; N_{11} \; = \; \{(4,7),3,(2,5),(8,10,6,9,11,12),13\}$$

$$S_{12}: \; N_{12} \; = \; \{(1,4,7,3),(2,5),(8,10,6,9,11,12),13\}$$

## 5. IMPLEMENTATION IN O(n log n) TIME

We have observed that optimal solutions to subproblems in the recursive computation can be represented as sets. In order to solve subproblems

with sets as inputs and outputs, we need to be able to perform the following operations efficiently: (i) identify a maximal or minimal element of a set, (ii) delete (a maximal or minimal) element from a set, (iii) form the union of two sets. Each of these operations is performed as many as $O(n)$ times. It follows that if we are to attain our goal of $O(n \log n)$ running time overall, we must be able to perform each of these operations in $O(\log n)$ steps.

There are data structures which facilitate these set operations known as "mergeable heaps". A complete exposition of these data structures can be found in the book of AHO, HOPCROFT and ULLMAN [2, p.152], with a demonstration that the desired running times can be attained.

## 6. RELATIONSHIP TO OPTIMAL LINEAR ORDERING

A problem related to sequencing is that of "optimal linear ordering", studied by ADOLPHSON and HU [1]. Let $G = (N,A)$ be an acyclic digraph and let $p_i'$ be a positive weight associated with node i and $q_{ij}$ a positive weight associated with arc(i,j).

The number $p_i'$ can be thought of as the "width" of node i. The number $q_{ij}$ can be thought of as the "number of wires" from i to j, where each wire leaves the "right" side of i and enters the "left" side of j. The object of the problem is to arrange the nodes adjacently in a one-dimensional array, in such a way that all wires are directed from left to right, and the sum of the wire lengths is minimized.

The sequencing problem and the optimal linear ordering problems are equivalent, under the relations

$$P_i = P_i'$$

$$w_i = \sum_j q_{ji} - \sum_j q_{ij}.$$

Thus, the algorithm presented in this paper can also be employed to solve the optimal linear ordering problem with series parallel precedence constraints.

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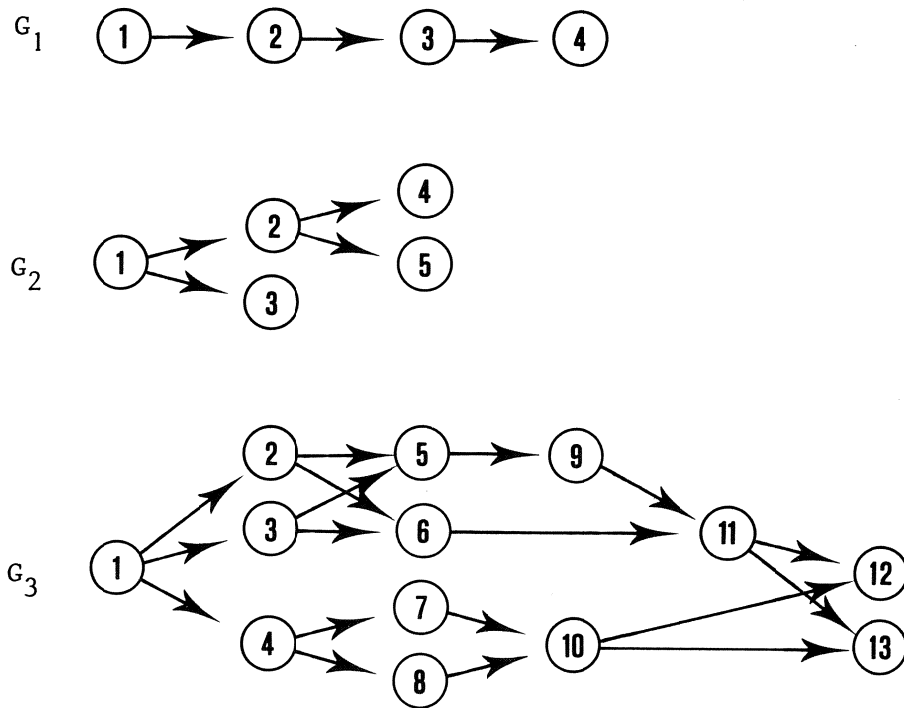[11]  SMITH, W.E., *"Various Optimizers for Single-Stage Production"*, Naval Res. Log. Quart. 3 (1956) p.59-66.

$G_1$   (1) $\longrightarrow$ (2) $\longrightarrow$ (3) $\longrightarrow$ (4)
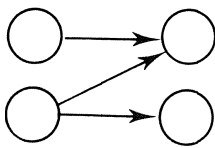
$G_2$

$G_3$

Figure 1 Series Parallel Digraphs.

Figure 2 A Nonseries Parallel Digraph.
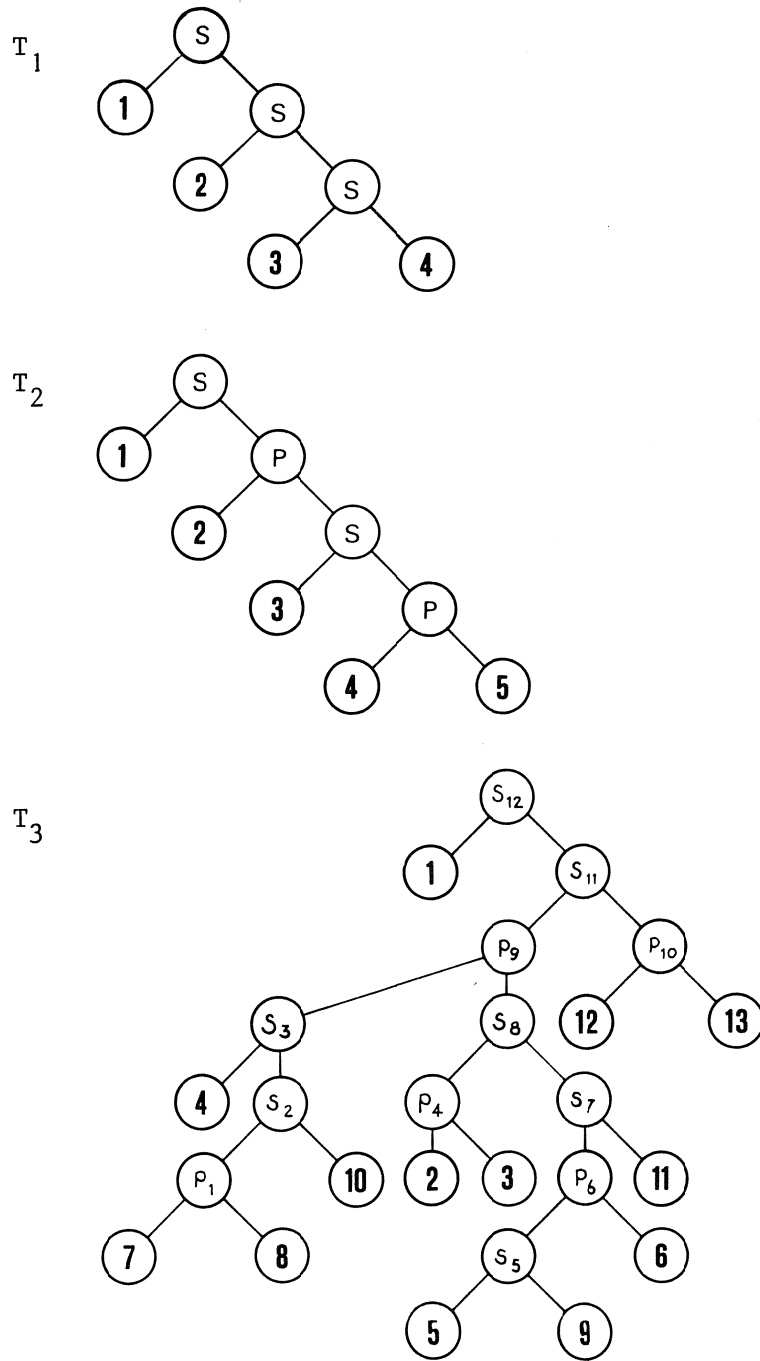
Figure 3 Decomposition Trees.